

# On Poisson geometry and supersymmetric sigma models

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## Abstract

By using the Poisson geometry, we develop a manifestly invariant and calculation-friendly formalism for handling  $UOSp(2|1)$ -supersymmetric field theories. In particular, the super-Lagrangians are written solely in terms of superfields, Poisson brackets and the moment map generating the  $UOSp(2|1)$  action. As an application of this formalism, we construct the Kalb-Ramond term for supersymmetric sigma models on the supersphere.

# 1 Introduction

Consider a smooth map  $\phi : \Sigma \rightarrow T$  where  $\Sigma$  and  $T$  are Riemannian manifolds. Such a map is called harmonic if it is a solution of field equations of the so called nonlinear sigma model associated to  $\Sigma$  and  $T$ . The Lagrangian of this model is given by the squared double norm (with respect to the metrics  $g$  on  $\Sigma$  and  $G$  on  $T$ ) of the derivation of  $\phi$  and the action is obtained by integration of the Lagrangian with respect to the measure  $d\mu_g$  on  $\Sigma$  induced by the metric  $g$ :

$$\mathcal{S}_G = \int d\mu_g ||d\phi||_{g,G}^2. \quad (1)$$

Obviously, the nonlinear sigma model is symmetric with respect to the group of isometries of the source manifold  $\Sigma$ . For example, if  $\Sigma$  is the two sphere  $S^2$  the model has the rotational  $SO(3)$  symmetry.

If  $\Sigma$  is two-dimensional, there exists a generalisation of the nonlinear sigma model considered mainly in string theory which is induced by a presence of an additional geometrical structure on the target  $T$ . This structure is called the Kalb-Ramond field [10] and it is nothing but a two-form field  $B$  on  $T$ . The pull-back  $\phi^*B$  integrated over  $\Sigma$  is then added to the original sigma model action in order to take into account the presence of  $B$ :

$$\mathcal{S}_{GB} = \int d\mu_g ||d\phi||_{g,G}^2 + \int \phi^*B. \quad (2)$$

Note that the Kalb-Ramond term  $\int \phi^*B$  is not only invariant with respect to the isometries of  $\Sigma$  but it is invariant even with respect to all diffeomorphisms of  $\Sigma$ .

If  $E$  is a Riemannian supermanifold with the bosonic body being the flat Euclidean 2-plane and odd coordinates being  $\xi, \bar{\xi}$  then there exists a supersymmetric generalisation of the nonlinear sigma model which includes the Kalb-Ramond term [6, 7]. Its action in the form of the Berezin integral reads

$$S_E = \int d\bar{z}dzd\bar{\xi}d\xi (G_{IJ}(Y^K) + iB_{IJ}(Y^K))DY^I\bar{D}Y^J, \quad (3)$$

where  $Y^I$  are the superfields (i.e. even functions on the Euclidean superplane) corresponding to the coordinates on the target space and  $G_{IJ}$  and  $B_{IJ}$  are respectively the components of the target space metric and the Kalb-Ramond field in those coordinates. Moreover, the supersymmetric covariant derivatives are defined as

$$\bar{D} := \partial_{\bar{\xi}} + \bar{\xi}\partial_{\bar{z}}, \quad D := \partial_{\xi} + \xi\partial_z. \quad (4)$$

Now the Kalb-Ramond part of the supersymmetric action (3) is a less geometric object as in the bosonic case since it can no longer be written in terms of a pull-back  $Y^*B$  by a sigma model superfield  $Y$ . Indeed, though  $B$  is the two-form, the volume form on the source supermanifold  $E$  is not a two-form due to the presence of odd differentials. As the result, the Kalb-Ramond term is not supersymmetric with respect to all superdiffeomorphisms of  $E$  but just with respect to the superisometries of  $E$ . In fact, the supersymmetric Kalb-Ramond term is *determined* by the

criterion of superinvariance with respect to the Euclidean superisometries and by the criterion that, when the superfields  $Y^I$  do not depend on the odd coordinates  $\bar{\xi}, \xi$ , the supersymmetric action (3) must reduce to the bosonic action (1).

The arguments of the previous paragraph show that in the supersymmetric case the Kalb-Ramond term must be determined case by case following what is the geometry of, possibly curved, world-sheet. More precisely, the Kalb-Ramond term is to be determined by two conditions: it must be invariant with respect to the superisometries of the worldsheet and it must reduce to the geometric term  $\int \phi^* B$  in the bosonic limit. In this paper, we shall consider sigma models on the so called supersphere [5] which is the simplest supersymmetrization of the standard sphere with the supergroup of superisometries being the unitary orthosymplectic supergroup  $UOSp(2|1)$ . In fact, to construct the action of the  $UOSp(2|1)$  supersymmetric non-linear sigma model on the curved super-worldsheet  $S^{2|2}$  is far from being just a straightforward generalisation of the flat super-Euclidean situation. This problem was posed already in [5] but it is fully solved only in the present article since the old work [5] and the subsequent work [18] constructed the supersymmetric sigma model on  $S^{2|2}$  without the Kalb-Ramond term.

Speaking more generally, string theoretical motivations caused recently a growing interest in formulations of rigidly supersymmetric field theories on curved space-times [1, 2, 3, 4, 8, 9, 11, 13, 14, 16, 17]. Obviously, the principal examples of such space-times are homogeneous spaces of supergroups e.g.  $S^{2|2}$  itself can be understood as the coset supermanifold  $UOSp(2|1)/U(1)$ . Although it is quite straightforward to construct various differential supersymmetric invariants of the rigid supersymmetry supergroup it is fairly less trivial task to work out which invariants give rise to viable field theories. Indeed, a seemingly "nice" invariant action principle written in the superfield formalism may be pathological when worked out in components. Typically, there may occur a violation of spin-statistics (a presence of quadratic bosonic derivatives in the fermionic kinetic term) and also other unwanted phenomena (like fourth order bosonic derivatives in the case the theory contains a gauge symmetry). Clues to select non-pathological candidates vary from case to case and no universal algorithms are available.

In the search of a viable definition of the supersymmetric sigma model on the super-worldsheet  $S^{2|2}$  we take a profit of the existence of a natural Poisson structure on  $S^{2|2}$ . In fact, the Poisson insights in the search for consistent supersymmetric field theories were not so far much explored in the literature though the original paper [5] using this approach is around already for some time. We believe that it's a pity because many expressions become less heavy when written in the Poisson way. Indeed, the principal result of this paper, which is the construction of the  $UOSp(2|1)$  invariant supersymmetric sigma model action including the Kalb-Ramond term on the supersphere, is obtained by relying on Poisson geometry both as the source of inspiration and as a technical tool. Without anticipating all details, this action reads

$$S_{sGB} = -\text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i\mathcal{M}B_{IJ}) \{\mathcal{M}^2, Y^I\} \{\mathcal{M}^2, Y^J\}, \quad (5)$$

where (the moment map)  $\mathcal{M}$  is a fixed  $uosp(2|1)$ -matrix valued function on  $S^{2|2}$ ,  $\{.,.\}$  stands for the Poisson bracket and  $d\mu_{S^{2|2}}$  is an  $uosp(2|1)$ -invariant measure on  $S^{2|2}$ . We note the appearance in the action (5) of the both moment map  $\mathcal{M}$ , which encompasses all supersymmetric generators and its square  $\mathcal{M}^2$ , which turns out to encompass all supersymmetric covariant

derivatives. It is precisely this circumstance that illustrates that from the structural point of view the supersymmetric action (5) is not quite a direct generalisation of the purely bosonic  $SO(3)$  invariant sigma model the action of which reads

$$S_{GB} = \frac{1}{2} \text{Tr} \int d\mu_{S^2} (G_{IJ} + iM B_{IJ}) \{M, Y^I\} \{M, Y^J\}. \quad (6)$$

Here the bosonic moment map  $M$  generates the  $SO(3)$  symmetry and its square does not appear in the story (in fact, unlike the moment map  $\mathcal{M}$ ,  $M$  squares to the unit matrix!). In spite of differences, the supersymmetric action (5) will be shown to reduce to the bosonic action (6) when the fermionic parts of the superfields  $Y^I$  are set to zero.

In a short Section 2, we review the concept of a Hermitian supermatrix and then in Section 3 we describe features of the Poisson geometries of the sphere  $S^2$  and of the supersphere  $S^{2|2}$ . Finally, in Section 4, we first construct the ordinary  $SO(3)$  invariant bosonic sigma model (6) on the ordinary sphere  $S^2$ , then the  $UOSp(2|1)$ -invariant super sigma model (5) on the supersphere  $S^{2|2}$  and we establish that in the absence of the fermions the superaction (5) does reduce to the bosonic action (6).

## 2 Supermatrices

Consider a complex Grassmann algebra  $G$  equipped with a  $\mathbb{C}$ -antilinear map call graded conjugation [15], which associates to every  $a \in G$  an element  $\bar{a} \in G$  in such a way that

$$\overline{ab} = \bar{a}\bar{b}, \quad \bar{\bar{a}} = (-1)^{p(a)}a, \quad a, b \in G. \quad (7)$$

Here  $p(a)$  means the Grassmann parity of  $a$ . By a supermatrix<sup>1</sup> we mean a square matrix  $M$  with a distinguished parities of indices for which the Grassmann parity of an element  $M_{ij} \in G$  is the same as the sum  $p(i) + p(j)$  of the index parities. Moreover, the elements  $M_{ij}$  of a "Hermitian supermatrix" satisfy the relation

$$M_{ij} = \bar{M}_{ji}, \quad i \geq j. \quad (8)$$

Note that in the purely bosonic case (8) remains true for all indices  $i, j$ , however in the supercase the restriction to the inequality  $i \geq j$  is essential.

The supertrace  $\text{STr}(M)$  of a supermatrix  $M$  is defined as

$$\text{STr}(M) := \sum_i (-1)^{p(i)} M_{ii}. \quad (9)$$

There is a natural supermeasure  $d\mu_H$  on the superhermitian matrices given by the formula

$$d\mu_H := \Pi_i dM_{ii} \Pi_{i < j} dM_{ij} d\bar{M}_{ij}, \quad (10)$$

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<sup>1</sup>In this paper we shall not consider "odd" supermatrices for which the Grassmann parity of an element  $M_{ij}$  is opposite to the sum  $p(i) + p(j)$  of the index parities

where  $dM_{ij}d\bar{M}_{ij}$  is the Berezin measure if  $p(i) + p(j)$  is odd and the standard Lebesgue measure on  $\mathbb{C} = \mathbb{R}^2$  if  $p(i) + p(j)$  is even.

The purely "bosonic" case corresponds to the situation where all index parities are even. All formulae presented in this section remains then true just the terminology flips e.g. from the supertrace  $\text{STr}$  to the ordinary trace  $\text{Tr}$  etc.

### 3 Sphere and supersphere

#### 3.1 Sphere $S^2$

We describe the ordinary two-dimensional sphere in the way best suited for the later supersymmetric generalisation. Thus we define the sphere  $S^2$  as the set of ordinary (purely bosonic) Hermitian  $2 \times 2$  matrices  $M$  such that

$$\text{Tr}(M) = 0, \quad \text{Tr}(M^2) = 2. \quad (11)$$

Indeed, in terms of the matrix components  $M_{ij}$  the first condition gives

$$M_{11} = -M_{22} \quad (12)$$

and the second one

$$M_{11}^2 + \bar{M}_{12}M_{12} = 1. \quad (13)$$

We shall view the Hermitian matrices  $M$  verifying the conditions (11) as points on the sphere but the matrix elements  $M_{ij}$  as particular *functions* on the sphere. The algebra  $C^{pol}(S^2)$  generated by  $M_{ij}$  is then a (dense) subspace of the space  $C^\infty(S^2)$  of smooth complex functions on  $S^2$ .

The sphere (e.g. viewed as the surface of the unit ball in the three-dimensional Euclidean space) can be naturally rotated by the group  $SO(3)$ . The infinitesimal generators  $V \in so(3)$  of this action turn out to act on the point  $M$  of the sphere as  $i[V, M]$ , where  $V$  is viewed as a traceless Hermitian matrix. (The Lie commutator on  $so(3)$  is then  $i$ -multiple of the matrix commutator.)

A natural Poisson bracket on  $C^{pol}(S^2)$  is defined by the following formula

$$\{\text{Tr}(UM), \text{Tr}(VM)\} := -i\text{Tr}([U, V]M), \quad (14)$$

where  $U, V$  are any constant traceless Hermitian matrices and  $[U, V]$  is the standard matrix commutator. The defining brackets (14) can be rewritten equivalently as

$$\{\text{Tr}(UM), M\} = i[U, M], \quad (15)$$

It is easy to verify that (14) indeed determines a Poisson bracket, in particular, the Poisson Jacobi identity is the consequence of the matrix Jacobi identity. Moreover, by taking the trace of (15) and of  $\{\text{Tr}(UM), M^2\} = i[U, M^2]$ , we derive

$$\{M, \text{Tr}(M)\} = \{M, \text{Tr}(M^2)\} = 0, \quad (16)$$

which is obviously needed for consistence with the definition (11) of the sphere.

Looking at (15), we immediately see that the  $so(3)$ -action is Hamiltonian with respect to the Poisson structure  $\{.,.\}$ . The corresponding moment map is clearly  $M$  and the Hamiltonian corresponding to the  $so(3)$  generator  $U$  is  $\text{Tr}(UM)$ . It follows that the Poisson structure (14) is  $so(3)$  invariant:

$$\{\text{Tr}(UM), \{f, g\}\} = \{\{\text{Tr}(UM), f\}g\} + \{f, \{\text{Tr}(UM), g\}\}, \quad \forall f, g \in C^{pol}(S^2). \quad (17)$$

A natural round measure on the sphere  $S^2$  can be defined with the help of the measure  $d\mu_H$  on Hermitian matrices weighted by delta functions of the constraints which define the sphere:

$$d\mu_{S^2} := d\mu_H \delta(\text{Tr}M) \delta\left(\frac{1}{2}\text{Tr}M^2 - 1\right) = dM_{11}dM_{12}d\bar{M}_{12}\delta(M_{11}^2 + \bar{M}_{12}M_{12} - 1). \quad (18)$$

We now wish to check, that this measure  $d\mu_{S^2}$  is indeed rotational invariant. For that it is sufficient to check the invariance of  $d\mu_H$  since the invariance of the arguments of the delta functions follows from (16). The infinitesimal change of coordinates induced by the rotation  $V$  is obviously

$$\delta M = i\varepsilon[V, M] \equiv i\varepsilon \text{Ad}_V M \quad (19)$$

where  $\varepsilon$  is a small parameter. The induced Jacobian is then

$$\det(1 + i\varepsilon \text{Ad}_V) = 1 + i\varepsilon \text{Tr}(\text{Ad}_V) = 1, \quad (20)$$

which means that the measure is indeed invariant.

The immediate consequence of the invariance of the measure  $d\mu_{S^2}$  is the formula

$$\int d\mu_{S^2} \{M, f\} = 0, \quad \forall f \in C^{pol}(S^2), \quad (21)$$

since  $\{M, f\}$  is the (matrix valued) variation of the function  $f$  under (all possible) infinitesimal rotations.

### 3.2 Supersphere $S^{2|2}$

A  $3 \times 3$  Hermitian supermatrix  $\mathcal{V}$  with two even indices 1, 2 and one odd index 3 is called orthosymplectic, if it satisfies

$$\mathcal{V}_{33} = 0, \quad \mathcal{V}_{23} = \bar{\mathcal{V}}_{13}. \quad (22)$$

An i-multiple of the standard commutator of two orthosymplectic supermatrices is again orthosymplectic and the corresponding (unitary orthosymplectic) Lie superalgebra is referred to as  $uosp(2|1)$ .

We now define the supersphere (or rather the algebra  $C^{pol}(S^{2|2})$  of polynomial functions on the supersphere) in a more invariant way than in [5], namely, we view it as the algebra generated by matrix elements of a Hermitian orthosymplectic supermatrix  $\mathcal{M}$  submitted to further constraints

$$\text{STr}(\mathcal{M}) = 0, \quad \text{STr}(\mathcal{M}^2) = 2. \quad (23)$$

Equivalently, solving the linear constraints give five independent generators which must verify the remaining quadratic constraint

$$\mathcal{M}_{11}^2 + \mathcal{M}_{12}\bar{\mathcal{M}}_{12} + 2\mathcal{M}_{13}\bar{\mathcal{M}}_{13} = 1. \quad (24)$$

Note, that if the odd generators  $\mathcal{M}_{13}, \bar{\mathcal{M}}_{13}$  vanish then (24) reduces to the defining relation (13) of the ordinary sphere.

The supersphere can be "superrotated" by the unitary orthosymplectic group  $UOSp(2|1)$  the Lie superalgebra of which is  $uosp(2|1)$ . Infinitesimal action of  $\mathcal{V} \in uosp(2|1)$  is just given by  $i[\mathcal{V}, \mathcal{M}]$ . This action is Hamiltonian (with the Hamiltonian equal to  $\text{STr}(\mathcal{V}\mathcal{M})$  and the moment map equal to  $\mathcal{M} \in uosp(2|1)$ ) if we define an  $uosp(2|1)$  invariant Poisson structure on  $C^{pol}(S^{2|2})$  by the bracket

$$\{\text{STr}(\mathcal{U}\mathcal{M}), \text{STr}(\mathcal{V}\mathcal{M})\} := -i\text{STr}([\mathcal{U}, \mathcal{V}]\mathcal{M}), \quad \mathcal{U}, \mathcal{V} \in uosp(2|1). \quad (25)$$

Indeed, (25) clearly implies

$$\{\text{STr}(\mathcal{V}\mathcal{M}), \mathcal{M}\} = i[\mathcal{V}, \mathcal{M}]. \quad (26)$$

It can be also easily checked that it holds

$$\{\mathcal{M}, \text{STr}(\mathcal{M})\} = \{\mathcal{M}, \text{STr}(\mathcal{M}^2)\} = 0 \quad (27)$$

as the consistency requires.

A natural  $uosp(2|1)$  invariant measure on the supersphere  $S^{2|2}$  can be defined with the help of the measure (10) on Hermitian supermatrices weighted by delta functions of all constraints which define the supersphere:

$$\begin{aligned} d\mu_{S^{2|2}} &:= d\mu_H \delta(\text{STr}\mathcal{M}) \delta\left(\frac{1}{2}\text{STr}\mathcal{M}^2 - 1\right) \delta(\mathcal{M}_{33}) \delta(\mathcal{M}_{23} - \bar{\mathcal{M}}_{13}) \delta(\bar{\mathcal{M}}_{23} + \mathcal{M}_{13}) = \\ &= d\mathcal{M}_{11} d\mathcal{M}_{12} d\bar{\mathcal{M}}_{12} d\mathcal{M}_{13} d\bar{\mathcal{M}}_{13} \delta(\mathcal{M}_{11}^2 + \mathcal{M}_{12}\bar{\mathcal{M}}_{12} + 2\mathcal{M}_{13}\bar{\mathcal{M}}_{13} - 1). \end{aligned} \quad (28)$$

Due to  $uosp(2|1)$  invariance of the constraints, in order to check the invariance of the measure  $d\mu_{S^{2|2}}$ , it is sufficient to check the invariance of  $d\mu_H$ . The infinitesimal change of coordinates induced by the  $uosp(2|1)$  element  $\mathcal{V}$  is obviously

$$\delta\mathcal{M} = i\varepsilon[\mathcal{V}, \mathcal{M}] \equiv i\varepsilon\text{Ad}_{\mathcal{V}}\mathcal{M} \quad (29)$$

where  $\varepsilon$  is a small parameter. The induced Berezinian is then

$$\text{sdet}(1 + i\varepsilon\text{Ad}_{\mathcal{V}}) = 1 + i\varepsilon\text{STr}(\text{Ad}_{\mathcal{V}}) = 1, \quad (30)$$

which means that the measure is indeed  $uosp(2|1)$  invariant.

The immediate consequence of the invariance of the measure  $d\mu_{S^{2|2}}$  is the formula

$$\int d\mu_{S^{2|2}} \{\mathcal{M}, f\} = 0, \quad \forall f \in C^{pol}(S^{2|2}), \quad (31)$$

since  $\{\mathcal{M}, f\}$  is the (matrix valued) variation of the function  $f$  under (all possible) infinitesimal  $uosp(2|1)$  transformations.

## 4 Sigma models

### 4.1 The bosonic case

Denote by  $y^I$ ,  $I = 1, \dots, n$  coordinates on the target Riemannian manifold  $T$  and, slightly abusing the notation, also the pull-backs  $\phi^* y^I$  by some smooth map  $\phi : S^2 \rightarrow T$ . The bosonic sigma model action  $\int d\mu_g ||d\phi||_{g,G}^2$  with the standard round metric  $g$  on  $S^2$  and a metric  $G_{IJ}(y^K)$  on  $T$  can be then rewritten in the following way (cf. Eq. (134) of [5]):

$$S_G = \int d\mu_{S^2} G_{IJ}(y^K) \{x_m, y^I\} \{x_m, y^J\}. \quad (32)$$

Here  $\{.,.\}$  is the Poisson structure defined in (14) and  $x_m$  are the fixed "Cartesian" functions on the sphere defined by the standard embedding of  $S^2$  into the Euclidean space  $\mathbb{R}^3$ . In the notation of Section 3.1 we have  $M = x_m \sigma^m$  (where  $\sigma^m$  are the standard Pauli matrices) and we can therefore rewrite (32) in more invariant way as

$$S_G = \frac{1}{2} \text{Tr} \int d\mu_{S^2} G_{IJ}(y^K) \{M, y^I\} \{M, y^J\}. \quad (33)$$

It may appear natural to add to (33) the Kalb-Ramond term in its most symmetric form  $\int \phi^* B$ , however, such expression does not lend itself to the supersymmetric generalisation. We shall instead rewrite the full sigma model action  $S_{GB} = \int d\mu_g ||d\phi||_{g,G}^2 + \int \phi^* B$  as

$$S_{GB} = \frac{1}{2} \text{Tr} \int d\mu_{S^2} (G_{IJ} + iM B_{IJ}) \{M, y^I\} \{M, y^J\} \equiv \text{Tr} \int d\mu_{S^2} L_{GB}, \quad (34)$$

where  $B_{IJ}$  are the components of the Kalb-Ramond form  $B$  in the coordinates  $y^I$ . The representation (34) of the full sigma model action was not obtained in [5], therefore we have to justify it. For that we must first check the  $so(3)$  invariance of the action (34) with respect to the infinitesimal rotations  $\delta_V y^I = \{\text{Tr}(VM), y^I\}$ . Using (15), we find successively

$$\delta_V G_{IJ}(y^K) = \{\text{Tr}(VM), G_{IJ}(y^K)\}, \quad (35)$$

$$\delta_V (B_{IJ}(y^K) M) = \{\text{Tr}(VM), B_{IJ}(y^K)\} M = \{\text{Tr}(VM), B_{IJ}(y^K) M\} - i[V, B_{IJ}(y^K) M], \quad (36)$$

$$\delta_V \{M, y^I\} = \{M, \delta_V y^I\} = \{\text{Tr}(VM), \{M, y^I\}\} - i[V, \{M, y^I\}], \quad (37)$$

$$\delta_V S_{GB} = \text{Tr} \int d\mu_{S^2} \{\text{Tr}(VM), L_{GB}\} - i \text{Tr} \int d\mu_{S^2} [V, L_{GB}] = 0. \quad (38)$$

Indeed, the last equality follows from (21) and from the fact that  $\text{Tr} V L_{GB} = \text{Tr} L_{GB} V$ . Now we are ready to verify that the Kalb-Ramond term  $\int \phi^* B$  can be written as

$$\int \phi^* B = \frac{1}{2} \text{Tr} \int d\mu_{S^2} iM B_{IJ}(y^K) \{M, y^I\} \{M, y^J\}. \quad (39)$$

Indeed the integral of the differential form  $\phi^*(dy^I \wedge dy^J)$  over  $S^2$  can be certainly written as  $\int d\mu_{S^2} \langle K, dy^I \wedge dy^J \rangle$  where  $K$  is some bivector on  $S^2$  (we write  $y^I$  instead of  $\phi^* y^I$ ). Because



$S^2$  is two-dimensional manifold, every two bivectors  $K$  and  $\tilde{K}$  are related as  $\tilde{K} = fK$ , where  $f$  is a function on the sphere. This means, in particular, that

$$\int \phi^* B = \frac{1}{2} \text{Tr} \int d\mu_{S^2} f i M B_{IJ}(y^K) \{M, y^I\} \{M, y^J\}. \quad (40)$$

However, the  $so(3)$  invariance of both  $\int \phi^* B$  and  $\text{Tr} \int d\mu_{S^2} i M B_{IJ}(y^K) \{M, y^I\} \{M, y^J\}$  means that  $f$  must be a  $so(3)$  invariant function hence a constant and it is easy to check that  $f = 1$ .

## 4.2 The supersymmetric case

In analogy with the bosonic case (34), it looks plausible that the action of the supersymmetric sigma model should be of the type

$$S_{tent} = \text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + i M B_{IJ}) \{\mathcal{M}, Y^I\} \{\mathcal{M}, Y^J\}. \quad (41)$$

Here  $Y^I$  are the sigma model superfields viewed as elements of  $C^{pol}(S^{2|2})$ , all other symbols were introduced in Section 3.2, and we wrote  $S_{tent}$  to indicate that this is just the tentative expression. Indeed, quite remarkably, this caution turns out to be fully justified since the tentative action (41) is pathological when worked out in components! (The problem is that upon the expansion in components the kinetic term for the fermions contains two bosonic derivatives.) It is therefore necessary to look for another expression and the main result of this article states that such a viable  $uosp(2|1)$ -invariant action is in fact given by the following formula

$$S_{sGB} = -\text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i M B_{IJ}) \{\mathcal{M}^2, Y^I\} \{\mathcal{M}^2, Y^J\}. \quad (42)$$

There are three things that we have to verify in order to show that the action (42) is indeed the correct one. First of all it is  $uosp(2|1)$ -invariance, then the correct bosonic limit and, thirdly, the absence of a quadratic expression in the bosonic derivatives in the fermionic part of the action.

1. The  $uosp(2|1)$ -invariance of (42) is verified in the exactly same way (35,36,38) as in the bosonic case. Only Eq. (37) has a slightly different supersymmetric counterpart:

$$\delta_{\mathcal{V}} \{\mathcal{M}^2, Y^I\} = \{\mathcal{M}^2, \delta_{\mathcal{V}} Y^I\} = \{\text{STr}(\mathcal{V} \mathcal{M}), \{\mathcal{M}^2, Y^I\}\} - i[\mathcal{V}, \{\mathcal{M}^2, Y^I\}]. \quad (43)$$

2. In order to speak about the bosonic limit of (42), we must first embed  $C^{pol}(S^2)$  in  $C^{pol}(S^{2|2})$  and then to consider the action (42) evaluated at the configurations  $\hat{y}^I \in C^{pol}(S^{2|2})$  which are the images of bosonic configurations  $y^I \in C^{pol}(S^2)$  upon this embedding. The embedding itself was constructed in [5] and it is completely defined by the images  $\hat{M}_{ij} \in C^{pol}(S^{2|2})$  (denoted by "hats") of the bosonic sphere generators  $M_{ij} \in C^{pol}(S^2)$ :

$$\hat{M}_{ij} = (\mathcal{M}_e(1 + \mathcal{M}_o^2))_{ij}, \quad i, j = 1, 2. \quad (44)$$

Here  $\mathcal{M}_e$  and  $\mathcal{M}_o$  are the even and the odd parts of the supermatrix  $\mathcal{M}$  and it is perhaps useful to rewrite (44) in components:

$$\hat{M}_{ij} = \mathcal{M}_{ij}(1 + \mathcal{M}_{13}\bar{\mathcal{M}}_{13}). \quad (45)$$

It is easy to check that (44) is consistent with the sphere and supersphere defining relations (11) and (23). Moreover, the embedding preserves the measure, i.e. it holds for every  $f \in C^{pol}(S^2)$ :

$$\int d\mu_{S^2} f = \int d\mu_{S^{2|2}} \hat{f}. \quad (46)$$

However, the Poisson structure is not completely preserved since it holds

$$\{\hat{f}, \hat{g}\}_{S^{2|2}} = \widehat{\{f, g\}_{S^2}} (1 + \mathcal{M}_{13} \bar{\mathcal{M}}_{13}), \quad f, g \in C^{pol}(S^2). \quad (47)$$

The crux of the argument is now based on the following identities which can be verified straightforwardly from the definition (25) of the supersphere Poisson structure:

$$\{\mathcal{M}_e^2, \hat{y}\} = \{\mathcal{M}_o^2, \hat{y}\} = 0, \quad \{\mathcal{M}_o \mathcal{M}_e, \hat{y}\} = \frac{1}{2} \mathcal{M}_o \{\mathcal{M}_e, \hat{y}\}, \quad \{\mathcal{M}_e \mathcal{M}_o, \hat{y}\} = \frac{1}{2} \{\mathcal{M}_e, \hat{y}\} \mathcal{M}_o, \quad (48)$$

where  $\hat{y} \in C^{pol}(S^{2|2})$  is the embedding of some  $y \in C^{pol}(S^2)$ . We can now start to evaluate the action  $S_{sGB}$  on the bosonic sigma model configuration  $y^K \in C^{pol}(S^2)$  embedded in  $C^{pol}(S^{2|2})$  as  $\hat{y}^K$ :

$$\begin{aligned} S_{sGB}(\hat{y}^K) &= -\text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i\mathcal{M}B_{IJ}) \{\mathcal{M}^2, \hat{y}^I\} \{\mathcal{M}^2, \hat{y}^J\} = \\ &= -\text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i\mathcal{M}B_{IJ}) \{\mathcal{M}_e \mathcal{M}_o + \mathcal{M}_o \mathcal{M}_e, \hat{y}^I\} \{\mathcal{M}_e \mathcal{M}_o + \mathcal{M}_o \mathcal{M}_e, \hat{y}^J\} = \\ &= -\frac{1}{4} \text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i\mathcal{M}_e B_{IJ}) (\mathcal{M}_o \{\mathcal{M}_e, \hat{y}^I\} \{\mathcal{M}_e, \hat{y}^J\} \mathcal{M}_o + \{\mathcal{M}_e, \hat{y}^I\} \mathcal{M}_o^2 \{\mathcal{M}_e, \hat{y}^J\}) = \\ &= -\frac{1}{2} \text{STr} \int d\mu_{S^{2|2}} \mathcal{M}_o^2 (G_{IJ} + i\hat{M}B_{IJ}) \{\hat{M}, \hat{y}^I\} \{\hat{M}, \hat{y}^J\} = \frac{1}{2} \text{Tr} \int d\mu_{S^2} (G_{IJ} + iMB_{IJ}) \{M, y^I\} \{M, y^J\}. \end{aligned} \quad (49)$$

In deriving (49) we have used (46), (47), the fact that  $\mathcal{M}_o \mathcal{M} \mathcal{M}_o = 0$ , that  $\mathcal{M}_o^2$  commutes with all matrices in the r.h.s. of (49) and also the facts like e.g.  $\{\mathcal{M}_e \mathcal{M}_o, \hat{y}^I\} \{\mathcal{M}_e \mathcal{M}_o, \hat{y}^J\}$  vanishes being the product of two odd upper-triangular matrices. Moreover, the last equality in (49) is obtained by integrating over the odd generators  $\mathcal{M}_{13}, \bar{\mathcal{M}}_{13}$  which are present only in the matrix  $\mathcal{M}_o^2$  since at every other place, including the measure delta function  $\delta(\mathcal{M}_{11}^2 + \mathcal{M}_{12} \bar{\mathcal{M}}_{12} + 2\mathcal{M}_{13} \bar{\mathcal{M}}_{13} - 1)$ , they are killed by the nilpotency.

3. It remains to verify the absence of a quadratic bosonic derivatives in the fermionic part of the action (42). For that we need not enter too far into the jungle of component calculations. We just consider the fermionic part  $Y_o^I$  of the superfield  $Y^I$  and we can write it as

$$Y_o^I(\mathcal{M}) = \Psi_-^I(\hat{M}) \mathcal{M}_{13} - \Psi_+^I(\hat{M}) \bar{\mathcal{M}}_{13}, \quad (50)$$

where the matrix  $\hat{M}$  was defined in (44) and the reality of the superfield  $Y_o^I(\mathcal{M})$  is ensured by requiring  $\bar{\Psi}_+ = \Psi_-$ . Now we study the expression  $\{\mathcal{M}^2, Y_o^I\} = \{\mathcal{M}^2, \Psi_-^I(\hat{M}) \mathcal{M}_{13} - \Psi_+^I(\hat{M}) \bar{\mathcal{M}}_{13}\}$  appearing in the action (42). We find from Eq. (14) that the components of the even part  $(\mathcal{M}^2)_e$  of the supermatrix  $\mathcal{M}^2$  Poisson-commute with the components of  $\mathcal{M}_e$ . This means that

only the Poisson bracket  $\{(\mathcal{M}^2)_o, Y_o^I\}$  can contain the bosonic derivatives  $\{\mathcal{M}_e, \Psi_\pm^I\}$ . By using Eq. (48), we obtain

$$\{(\mathcal{M}^2)_o, \Psi_\pm^I(\hat{M})\} = \{\mathcal{M}_o \mathcal{M}_e + \mathcal{M}_e \mathcal{M}_o, \Psi_\pm^I(\hat{M})\} = \frac{1}{2} \mathcal{M}_o \{\mathcal{M}_e, \Psi_\pm^I\} - \frac{1}{2} \{\mathcal{M}_e, \Psi_\pm^I\} \mathcal{M}_o \quad (51)$$

hence

$$\{(\mathcal{M}^2)_o, Y_o^I\} = -\Psi_-^I(\hat{M})\{(\mathcal{M}^2)_o, \mathcal{M}_{13}\} + \Psi_+^I(\hat{M})\{(\mathcal{M}^2)_o, \bar{\mathcal{M}}_{13}\} + \mathcal{M}_{13} \bar{\mathcal{M}}_{13} V \quad (52)$$

This means that the bosonic derivatives of fermions  $\{\mathcal{M}_e, \Psi_\pm^I\}$  appear only in the expression  $V$  that multiplies  $\mathcal{M}_{13} \bar{\mathcal{M}}_{13}$ . The fact that  $\mathcal{M}_{13} \bar{\mathcal{M}}_{13}$  squares to zero thus excludes a presence of the quadratic bosonic derivatives in the fermionic part of the action.

### 4.3 Supersymmetric sigma model on $S^{2|2}$ in components

This paper would not be complet, if we did not present the component action derived from the superfield action (42) via the ansatz

$$Y^I(\mathcal{M}) = \hat{y}^I + \Psi_-^I \mathcal{M}_{13} - \Psi_+^I \bar{\mathcal{M}}_{13} + F^I \mathcal{M}_{13} \bar{\mathcal{M}}_{13}. \quad (53)$$

By eliminating the auxiliary fields  $F^I$ , we obtain

$$\begin{aligned} S_{sGB} = & \frac{1}{2} \int d\mu_{S^2} \text{Tr} \left[ (G_{IJ} + iMB_{IJ}) \{M, y^I\} \{M, y^J\} + 2G_{IJ} (i\{M, \Psi^J\} + \Psi^J) \bar{\Psi}^I \right] + \\ & + \int d\mu_{S^2} \left[ \bar{\Psi}^I i\{M, y^K\} (\Gamma_{IKL} + iMH_{IKL}) \Psi^L - \frac{1}{8} \mathcal{R}_{IJKL} (\bar{\Psi}^I \Psi^K - \bar{\Psi}^I M \Psi^K) (\bar{\Psi}^J \Psi^L - \bar{\Psi}^J M \Psi^L) \right], \end{aligned} \quad (54)$$

where

$$\Psi^I := \begin{pmatrix} \Psi_+^I \\ \Psi_-^I \end{pmatrix}, \quad \bar{\Psi}^I := (\bar{\Psi}_+^I \quad \bar{\Psi}_-^I) = (\Psi_-^I \quad -\Psi_+^I) \quad (55)$$

and the notation  $\{M, \Psi^J\}$  means at the same time the Poisson bracket and the matrix action on a column vector:

$$\{M, \Psi\}_\alpha := \sum_\beta \{M_{\alpha\beta}, \Psi_\beta\}. \quad (56)$$

Moreover, the quantities  $\Gamma_{IKL}$ ,  $H_{IKL}$  and  $\mathcal{R}_{IJKL}$  are defined as

$$\Gamma_{IKL} = \frac{1}{2} (\partial_L G_{KI} + \partial_K G_{IL} - \partial_I G_{KL}), \quad H_{IKL} = \frac{1}{2} (\partial_I B_{KL} + \partial_K B_{LI} + \partial_L B_{IK}),$$

$$\mathcal{R}_{IJKL} := G_{IM} \mathcal{R}_{JKL}^M, \quad \mathcal{R}_{JKL}^M := \partial_K E_{LJ}^M - \partial_L E_{KJ}^M + E_{KN}^M E_{LJ}^N - E_{LN}^M E_{KJ}^N, \quad (57)$$

$$E_{LJ}^K := G^{KN} (\Gamma_{NLJ} + iH_{NLJ}). \quad (58)$$

We recognize in the quantity  $G^{KN} \Gamma_{NLJ}$  the standard Christoffel symbol corresponding to the metric  $G_{IJ}$ , the totally antisymmetric tensor  $H_{IJK}$  is nothing but the exterior derivative of

the two-form  $B_{IJ}$  and  $\mathcal{R}_{IJKL}$  are the components of the modified Riemann curvature tensor corresponding to the connection  $E_{LJ}^K$  containing the torsion part  $H_{LJ}^K$ .

We notice that the component action (54) has again the elegant property that, apart from the dynamical fields  $y^I$  and  $\Psi^I$ , it contains just the Poisson brackets and the moment map  $M$ . However, we have to admit that in this particular case we were not able to preserve the elegance in all intermediate calculations. Indeed, while everything else in this paper was computed very directly and effortlessly thanks to our invariant Poisson language, the formula (54) was worked out by a tedious component calculation.

## 5 Conclusions and outlook

Apart from our main result which is the construction of the action of the  $UOSp(2|1)$  supersymmetric sigma model with the Kalb-Ramond term, the present article also offers a conceptual simplification and a technical streamlining of the results of the reference [5]. In particular, all five generators  $R_3, R_\pm, V_\pm$  of the Lie superalgebra  $uosp(2|1)$  and all three supersymmetric covariant derivatives  $\Gamma, D_\pm$  appearing explicitly in majority of formulas of [5] are conveniently arranged as matrix elements of a single supermatrix  $\mathcal{M}$  and its square  $\mathcal{M}^2$ . Moreover, all calculations of [5] can be rephrased in terms of the matrices  $\mathcal{M}$  and  $\mathcal{M}^2$  as a whole without a necessity to manipulate the matrix elements themselves. As for the outlook, we expect that the construction of supersymmetric gauge theories on the supersphere presented in [12] could be equally streamlined and rendered conceptually more transparent by using the moment map supermatrix  $\mathcal{M}$ .

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